

## On the operation of a tunable electromagnetic detector for gravitational waves

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1978 J. Phys. A: Math. Gen. 11 1949

(<http://iopscience.iop.org/0305-4470/11/10/013>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 30/05/2010 at 15:11

Please note that [terms and conditions apply](#).

# On the operation of a tunable electromagnetic detector for gravitational waves

F Pegoraro<sup>†</sup>, E Picasso<sup>‡</sup> and L A Radicati<sup>‡§</sup>

<sup>†</sup>Scuola Normale Superiore, Pisa, Italy

<sup>‡</sup>CERN, Geneva, Switzerland

Received 6 December 1977, in final form 20 April 1978

**Abstract.** The interaction of a gravitational wave with an oscillating EM field inside a detector is analysed in the frame in which the detector's walls are at rest. The interaction is described in terms of a time-dependent dielectric tensor and the energy transfer is derived from a generalised Poynting theorem. The energy transfer is maximum when the GW frequency matches one of the frequencies of quadrupole moment of the EM field. Provided the  $Q$ -value of the detector is sufficiently high this method seems particularly suitable for observing GW's emitted by periodic sources.

## 1. Introduction

The first experimental attempt to detect gravitational waves (GW) (Weber 1969) was sensitive to frequencies of the order of 1 kHz which are supposed to be present only in the bursts of gravitational energy emitted during supernova collapses. Essentially the same frequency range has been investigated in subsequent experiments conducted in the Soviet Union (Braginskii *et al* 1972), in the United Kingdom (Drever *et al* 1973), in the United States (Tyson 1973, Levine and Garwin 1973), in Western Europe (Bramanti and Maischberger 1972, Billing *et al* 1975), and in Japan (Hirakawa and Narihara 1975).

The detectors used in all these experiments are mechanical quadrupole antennas which can resonate at the gravitational frequency. The power absorbed is proportional to the gravitational energy flux. Other types of detectors, based on the interaction between the GW and the electromagnetic field, have been suggested by Braginskii and Menskii (1971).<sup>¶</sup> The power absorbed by them is proportional to the gravitational energy flux, or to its square root, depending on the initial state of the EM field (Braginskii *et al* 1974). The obvious advantage of the electromagnetic detectors is that they can easily be tuned on a wider range of frequencies than the mechanical ones and thus be made to respond either to the relatively high frequencies present in the gravitational bursts emitted during stellar collapses or to the low frequencies emitted by continuous periodic sources such as rotating neutron stars and close binary systems. The fluxes in the latter case are considerably smaller than those expected from stellar collapses but, since the emission is continuous, one can take advantage of the possibility of repeating the experiment to reduce the noise-to-signal ratio.

<sup>§</sup> Permanent address: Scuola Normale Superiore, 56100 Pisa, Italy.

<sup>¶</sup> The interaction between the EM field and a static gravitational field has recently been discussed by Braginskii *et al* (1977) in connection with post-Newtonian effects.

To compare different sources it is convenient to define the dimensionless quantity†

$$h = \frac{1}{\Omega} (32\pi GI)^{1/2} \quad (1.1)$$

where  $G$  is the gravitational constant,  $I$  is the energy flux and  $\Omega$  the angular frequency of the GW. Another way of writing  $h$  is

$$h = GM_{\text{eq}}/r \quad (1.2)$$

i.e. as the gravitational potential produced by an equivalent mass at the distance  $r$  of the source. The equivalent mass  $M_{\text{eq}}$  is given in terms of the quadrupole moment  $D$  of the source by

$$M_{\text{eq}} = \frac{2}{3} \left( \frac{2}{5} \right)^{1/2} D \Omega^2. \quad (1.3)$$

The values of  $h$  and  $I$  for a few typical sources are given in table 1 (see, e.g., Press and Thorne 1972, Drever 1977, Amaldi and Pizzella 1978).

**Table 1.**

Source	$h$	$\Omega(\text{rad s}^{-1})$	$I(\text{erg cm}^{-2} \text{s}^{-1})$
Extragalactic supernovae	$2 \times 10^{-22} - 10^{-20}$	$10^3 - 10^5$	$10^4 - 10$
Crab pulsar	$10^{-24}$	377	$3 \times 10^{-7}$
Iota Bootes	$6 \times 10^{-21}$	$5.4 \times 10^{-4}$	$1.8 \times 10^{-10}$
HZ 29	$3.6 \times 10^{-22}$	$1.2 \times 10^{-2}$	$3 \times 10^{-10}$

The maximum value of  $h$  for the Crab pulsar corresponds to the implausible assumption that the gravitational braking be of the same order as the electromagnetic braking. The actual value can be several orders of magnitude smaller. The two binaries in table 1 have  $h$ 's of the same order of magnitude as the extragalactic supernovae, but frequencies at least five orders of magnitude smaller.

In the following sections we shall analyse the possibility of detecting these GW's through their effect on the electromagnetic field enclosed inside the detector. We will describe the change of the EM field due to the combined effect of the modification of the boundary conditions and of the direct interaction between the EM field and the GW's in terms of a dielectric and magnetic susceptibility tensor  $\epsilon$  which is related to the metric tensor. Moreover, we shall concentrate on the detectors in which the power exchanged between the detector and the GW is proportional to the square root of the incoming gravitational flux. It will be shown that the power exchanged is proportional to the EM energy  $W_0$  in the detector and is given by:

$$P = \Lambda \text{tr}(\dot{a}\Pi)W_0. \quad (1.4)$$

Here  $\Lambda$  is a dimensionless coefficient which depends upon the mechanical properties of the detector,  $a$  is the correction to the metric tensor due to the GW in the transverse traceless gauge,  $\Pi$  is the dimensionless quadrupole moment associated with the configuration of the EM field and the dot denotes time derivative.

† We shall set the velocity of light equal to one.

The configurations of the detectors that we will discuss consist of two arms at right angles between which the energy oscillates at the frequency of the GW,  $\Omega$ . In this way we obtain a resonance between the frequency of the EM energy and the time variation of  $a$ . Since  $\dot{a}$  is proportional to  $h\Omega$ , the leading contribution to the energy absorbed by the detector during time  $\tau$  is

$$\Delta W \sim \Lambda W_0 h \Omega \tau \tag{1.5}$$

where the maximum value of  $\tau$  is of the order of the detector's relaxation time.

Equations (1.4) and (1.5) are valid provided  $P \ll IL^2$  where  $L$  is the characteristic size of the detector. This condition is easily satisfied in all cases we shall consider.

In the simplified case when the mechanical properties of the detector are described in terms of a single frequency  $\omega_m$  and a relaxation time  $1/(2\gamma_m)$  the coefficient  $\Lambda$  of equations (1.4) and (1.5) becomes

$$\Lambda \approx \frac{\Omega^2}{\Omega^2 - \omega_m^2} \quad \text{for } |\Omega - \omega_m| > \gamma_m \tag{1.6}$$

or

$$\Lambda = \frac{\omega_m}{\gamma_m} = Q_m \quad \text{for } \Omega = \omega_m. \tag{1.7}$$

In writing these relations we have neglected terms of order  $L^2\Omega^2$  which, for all the detectors we shall discuss, are negligibly small. As one can see from equation (1.6), the case of rigid walls, where  $\omega_m \gg \Omega$ , leads to a strong reduction of the power absorbed.

## 2. Power exchange

### 2.1. The rest frame

The power exchanged between the GW and the EM field inside a detector (equation (1.4)) can be split into two terms: a *direct* one arising from the interaction of the GW with the EM field, and an *indirect* one due to the effect which the motion of the detector's walls has on the EM field through the change of the boundary conditions. The indirect interaction vanishes in the coordinate system in which the walls are at rest (rest frame). This frame depends in a complicated way upon the geometrical structure and the mechanical properties of the detector. We will only consider the simplified case in which the detector's walls are treated as a system of equal particles elastically bound to the centre of mass.

We shall analyse the interaction between the GW and the EM field in the rest frame of the detector to first order in the amplitude  $h$  of the GW. We shall assume that the size of the detector is small compared to the wavelength of the GW and neglect terms of order  $h(\Omega L)^2$  in the metric tensor in the rest frame. The relation between the definition of the rest frame and the mechanical properties of the detector is derived in the appendix together with the corrections due to the finite size of the detector.

Consider a coordinate system  $x^\mu(\lambda)$  depending on a parameter  $\lambda$  determined in such a way as to transform away the motion of the detector's walls induced by the GW. This frame satisfies the following conditions:

- (i) In the absence of the GW it reduces to the same Minkowski frame.
- (ii) The origin coincides with the centre of mass of the detector.
- (iii) The wavevector  $\mathbf{k}_\mu$  of the GW is  $\mathbf{k}_\mu = (k_0, 0, 0, -k_3)$ , where  $k_0 = k_3 = \Omega$  is the GW's frequency.
- (iv) The metric tensor  $g_{\mu\nu}(\lambda)$  is of the form†

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(\lambda) \tag{2.1}$$

where  $\eta_{\mu\nu}$  is the Minkowski tensor and  $h_{\mu\nu}(\lambda)$ , which represents the contribution of the GW, is of the form

$$h_{\mu\nu}(\lambda) = (2\lambda - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O(\hbar\Omega^2L^2) \tag{2.2}$$

where

$$a_{ij} = \hbar \operatorname{Re} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \quad a_{ij} = a_{ji}, \quad \operatorname{Tr} a = 0 \tag{2.3}$$

with  $i, j = 1, 2$  and

$$\begin{aligned} \alpha &= \alpha_0 \exp i[\Omega(x^3 - x^0) + \theta_\alpha] \\ \beta &= \beta_0 \exp i[\Omega(x^3 - x^0) + \theta_\beta] \\ \alpha_0, \beta_0 &\geq 0, \quad \alpha_0^2 + \beta_0^2 = 1. \end{aligned} \tag{2.4}$$

In the simplified case when the detector can be treated as a system of oscillators with frequency  $\omega_m$  and relaxation time  $\tau_m = \frac{1}{2}\gamma_m$ , the value of  $\lambda$  which corresponds to the rest frame of the detector is given by (see appendix)

$$\lambda = \frac{1}{2}\omega_m^2 / (\omega_m^2 - \Omega^2) \tag{2.5}$$

if  $\omega_m \neq \Omega$  and by

$$|\lambda| \approx \omega_m / \gamma_m = Q_m \tag{2.6}$$

if the resonance condition  $\omega_m = \Omega$  is met. In the latter case the phases  $\theta_\alpha$  and  $\theta_\beta$  in equation (2.4) are shifted by  $\pi/2$ .

### 2.2. The dielectric tensor

As is well known a GW propagating in vacuum induces a time-dependent dielectric tensor proportional to its amplitude. This dielectric tensor depends upon the reference frame and we will calculate it in the rest frame. In this frame the indirect interaction vanishes and the effect of the GW on the EM field is entirely taken into account by the dielectric tensor.

† We shall use the sign convention (+, -, -, -); Greek indices run from 0 to 3, Latin indices from 1 to 2; summation over repeated indices is implied.

Let us define as usual (Landau and Lifshitz 1962) the electric field  $\mathbf{E} = (E_1, E_2, E_3)$  and magnetic flux density  $\mathbf{B} = (B_1, B_2, B_3)$  by

$$\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} = F_{\mu\nu} \quad (2.7)$$

where  $F_{\mu\nu}$  is the EM field tensor. The displacement  $\mathbf{D} = (D^1, D^2, D^3)$  and the magnetic field  $\mathbf{H} = (H^1, H^2, H^3)$  are defined by

$$\sqrt{g_{00}}F^{\mu\nu} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & H^3 & -H^2 \\ D^2 & -H^3 & 0 & H^1 \\ D^3 & H^2 & -H^1 & 0 \end{pmatrix} = \sqrt{g_{00}}g^{\mu\alpha}F_{\alpha\beta}g^{\beta\nu}. \quad (2.8)$$

From (2.7), (2.8) and (2.2) it follows that

$$D^i = \epsilon^{ij}E_j, \quad D^3 = E_3, \quad B_i = \mu_{ij}H^j, \quad B_3 = H^3 \quad (2.9)$$

where  $i, j = 1, 2$  and

$$\begin{aligned} \epsilon^{ij} &= -[\eta^{ij} + (1 - 2\lambda)a^{ij}] \\ \mu_{ij} &= -[\eta_{ij} + (1 - 2\lambda)a_{ij}]. \end{aligned} \quad (2.10)$$

The value of each component of the dielectric tensor  $\epsilon^{ij}$  is equal to the corresponding component of the magnetic permeability tensor  $\mu_{ij}$ . The corrections to equations (2.9) and (2.10) which are introduced by the finite size of the detector are given in the appendix.

Equations (2.10) show that the dielectric tensor in the rest frame depends upon the mechanical properties of the detector: in the  $\lambda = \frac{1}{2}$  frame it reduces to the identity, whereas in the free-falling frame ( $\lambda = 0$ ) we have

$$\epsilon^{ij} = -(\eta^{ij} + a^{ij}). \quad (2.11)$$

In the case corresponding to equation (2.6) ( $\Omega = \omega_m$ ), the dielectric tensor becomes

$$\epsilon^{ij} = -[\eta^{ij} + 2Q_m a^{ij}(\theta_\alpha - \pi/2, \theta_\beta - \pi/2)]. \quad (2.12)$$

The correction to the vacuum value of the dielectric tensor,  $-\eta^{ij}$ , reduces in this case to the product of the mechanical quality factor  $Q_m$  of the detector times the amplitude of the GW. Due to the resonance, the tensor  $a^{ij}$  appears in equations (2.12) with phases shifted by  $\pi/2$  with respect to the phases  $\theta_\alpha$  and  $\theta_\beta$  of the GW.

### 2.3. The Poynting theorem

The fields  $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$  defined in § 2.2 satisfy the standard Maxwell equations (with  $\mathbf{x} = (x^1, x^2, x^3)$ ,  $t = x^0$ ). From them one derives the Poynting theorem in the usual way:

$$\text{div}(\mathbf{E} \times \mathbf{H}) + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) = -\frac{1}{2} \text{Tr}(\dot{\epsilon}T) - 4\pi \mathbf{j} \cdot \mathbf{E}. \quad (2.13)$$

Here  $\mathbf{j} = (j^1, j^2, j^3)$  is the current density,  $T_{ij}$  are the components of the energy-momentum tensor in the (1-2) plane and

$$\begin{aligned} \text{Tr}(\epsilon T) &= \epsilon^{ij} T_{ij} \\ &= -(1-2\lambda) \dot{a}^{ij} T_{ij} \\ &= -\frac{1-2\lambda}{8\pi} [\dot{a}_{11}(B_1^2 + E_1^2 - B_2^2 - E_2^2) + 2\dot{a}_{12}(B_1 B_2 + E_1 E_2)]. \end{aligned} \quad (2.14)$$

In deriving equation (2.14) we have made use of equations (2.10) which, as shown in the appendix, are correct up to terms of order  $h(\Omega L)^2$ .

To derive the power exchanged between the GW and the detector we integrate equation (2.13) over the volume of the detector. Assuming that  $\mathbf{E} \times \mathbf{B}$  vanishes on the walls, we obtain

$$dW/dt = W \text{Tr}(\dot{\epsilon} \Pi) - \gamma_{EM} W \quad (2.15)$$

where

$$W = \frac{1}{8\pi} \int d^3 \mathbf{x} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \quad (2.16)$$

is the EM energy inside the detector;

$$\Pi_{11} = -\Pi_{22} = \frac{1}{2W} \int d^3 \mathbf{x} (B_1^2 + E_1^2 - B_2^2 - E_2^2) \quad (2.17)$$

$$\Pi_{12} = \Pi_{21} = \frac{1}{W} \int d^3 \mathbf{x} (B_1 B_2 + E_1 E_2)$$

and

$$\gamma_{EM} = \frac{2}{W} \int d^3 \mathbf{x} \mathbf{j} \cdot \mathbf{E}. \quad (2.18)$$

Equation (2.15) has been obtained by neglecting terms of order  $h\Omega L$  due to the  $x^3$  dependence of  $\epsilon$ .

The tensor  $\Pi$ , which in general depends upon time, represents the (dimensionless) quadrupole moment of the EM field in the detector, and  $1/\gamma_{EM}$  is the EM relaxation time of the detector.

The second term in the right-hand side of equation (2.15), which is due to the losses of the detector, is, in all realistic cases, much larger than the first one which arises from the interaction with the GW.

Integrating equation (2.15) we get

$$W(t) = W_0 \left( 1 + \int_0^t \text{Tr}(\dot{\epsilon}(t') \Pi(t')) dt' \right) \exp(-\gamma_{EM} t) \quad (2.19)$$

where  $W_0$  is the EM energy in the detector at  $t=0$ . The second term in the large parentheses, which is correct only to first order in  $h\Omega/\gamma_{EM} \ll 1$ , depends upon the intensity, the polarisation, the phases and the frequency of the GW, through  $\dot{\epsilon}$  and upon the geometry of the detector through the quadrupole moment  $\Pi$ . This term changes sign when the configuration of the detector is rotated by  $\pi/2$  around the  $x^3$  axis. This behaviour reflects the rotational properties of the spin-two carried by the GW.

Let us define

$$\Delta W(t) = W(t, h) - W(t, 0) = W_0 \exp(-\gamma_{EM}t) \int_0^t \text{Tr}(\dot{\epsilon}(t')\Pi(t')) dt' \tag{2.20}$$

which represents the difference between the energy in the detector at time  $t$  with and without the GW.

For  $t > 1/\Omega$  the integral in the right-hand side of equation (2.20) is largest when the frequency  $\Omega$  of the GW coincides with one of the frequencies of the quadrupole moment  $\Pi$ . In this case  $\Delta W$  can be written in the form

$$\Delta W(t) = W_0(1 - 2\lambda)h\Omega \left( \rho t + \frac{1}{\Omega} f(t) \right) \exp(-\gamma_{EM}t) \tag{2.21}$$

where

$$\rho = \lim_{T \rightarrow \infty} \frac{1}{2Th\Omega} \int_0^T \text{Tr}(\dot{a}(t')\Pi(t')) dt' \tag{2.22}$$

is a parameter which measures the efficiency of the detector and  $f(t)$  is an oscillating function. For  $t \sim 1/\gamma_{EM}$  the oscillating function can be neglected provided

$$|f| < \frac{\Omega\rho}{\gamma_{EM}} \tag{2.23}$$

and the maximum value of  $\Delta W$  becomes:

$$\Delta W = h \frac{\Omega}{\gamma_{EM}} W_0\rho(1 - 2\lambda). \tag{2.24}$$

To determine the explicit form of  $\rho$  let

$$\Pi_\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \cos(\Omega t)\Pi(t) \tag{2.25}$$

be the Fourier component of  $\Pi$  at the frequency of the GW. From equation (2.22) it follows that:

$$\rho = |\Pi_\Omega|g \tag{2.26}$$

where

$$|\Pi_\Omega| = (\frac{1}{2} \text{Tr}\Pi_\Omega^2)^{1/2} \leq 1 \tag{2.27}$$

is the invariant 'length' of  $\Pi_\Omega$  and  $g$  ( $|g| \leq 1$ ) is a function of the phases and the polarisations of the GW.

We remark that the energy transfer given in equation (2.21) represents the combined effect of the direct and indirect interactions between the GW and the detector. As we have already stated these two contributions are entirely described, in the rest frame, by the time variation of the dielectric constant.

### 3. Resonant detectors

In this section we shall discuss two types of resonant EM detectors: a low-frequency detector where the EM frequency is equal to  $\Omega/2$ , and a high-frequency one, in which the EM field oscillates at two frequencies  $\omega_+, \omega_- \gg \Omega$ , such that  $\omega_+ - \omega_- = \Omega$ .



Our first example is a LC circuit oscillating at frequency  $\omega$ . To simplify the analysis we assume that EM energy and the quadrupole moment can be calculated with sufficient approximation from the following field configuration:

$$\begin{aligned} \mathbf{E} &= E_0 \cos \omega t (\cos \phi_E, \sin \phi_E, 0) \\ \mathbf{B} &= B_0 \sin \omega t (\cos \phi_B, \sin \phi_B, 0) \end{aligned} \tag{3.1}$$

where  $E_0$  and  $B_0$  are constant inside the volumes  $V_E$  and  $V_B$  of the capacitor and of the coil respectively, and vanish outside them. They satisfy the condition

$$E_0^2/B_0^2 = V_B/V_E \tag{3.2}$$

which expresses the equality of the time average of the electric and magnetic energies. The quadrupole moment of the field configuration specified by equation (3.1) is the sum of two terms, one constant in time and the other oscillating at frequency  $2\omega$ :

$$\Pi(t) = \Pi_0 + \Pi_{2\omega} \cos 2\omega t. \tag{3.3}$$

To maximise the energy exchange between the EM circuit and the GW,  $\Pi(t)$  must oscillate at the same frequency as  $\epsilon(t)$ , i.e. we must have  $\omega = \Omega/2$ .

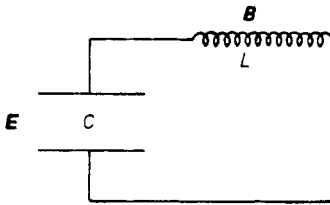
One can easily show that the value of the invariant length of  $\Pi_{2\omega}$  for this simplified configuration is

$$|\Pi_{2\omega}| = |\sin(\phi_E - \phi_B)|. \tag{3.4}$$

Similarly the invariant length of  $\Pi_0$  is

$$|\Pi_0| = |\cos(\phi_E - \phi_B)| \tag{3.5}$$

The maximum efficiency  $\rho$  therefore obtains when  $\phi_E - \phi_B = \pm \pi/2$ , i.e. when the electric field in the capacitor is perpendicular to the magnetic field in the coil. In this case  $|\Pi_0| = 0$  (see figure 1).



**Figure 1.** Low-frequency LC resonator.

In a more realistic case, the electric and magnetic fields of an oscillating circuit cannot be written in the simple form (3.1). However equations (3.4) and (3.5) remain true if  $E_3 = B_3 = 0$ , with  $\phi_E$  and  $\phi_B$  defined by

$$\cos 2\phi_E = \frac{\int d^3 \mathbf{x} (E_1^2 - E_2^2)}{\int d^3 \mathbf{x} (E_1^2 + E_2^2)} \quad \cos 2\phi_B = \frac{\int d^3 \mathbf{x} (B_1^2 - B_2^2)}{\int d^3 \mathbf{x} (B_1^2 + B_2^2)} \tag{3.6}$$

For low-frequency GW's the resonance condition  $\omega = \Omega/2$  requires large inductances and capacities which make this conceptually simple detector not very promising in practice. This difficulty can be avoided by making the GW resonate with the beat frequency of two coupled LC circuits operating at high frequencies and therefore of manageable sizes.

Let  $\omega_+$  and  $\omega_-$ ,  $\omega_+ > \omega_-$  be the two eigenfrequencies of the two coupled oscillators and let

$$2\omega = \omega_+ + \omega_-, \quad 2\eta\omega = \omega_+ - \omega_- \quad (3.7)$$

with  $\eta < 1$ . We will distinguish with superscripts 1 and 2 the quantities referring to each circuit which we shall assume to be spatially separated. As in the previous case we will calculate the quadrupole moment with a simplified field configuration, namely:

$$\begin{aligned} \mathbf{E}^{(1)} &= E_0^{(1)} \cos \omega t \cos \eta\omega t (\cos \phi_E^{(1)}, \sin \phi_E^{(1)}, 0) \\ \mathbf{E}^{(2)} &= E_0^{(2)} \sin \omega t \sin \eta\omega t (\cos \phi_E^{(2)}, \sin \phi_E^{(2)}, 0) \\ \mathbf{B}^{(1)} &= B_0^{(1)} \sin \omega t \cos \eta\omega t (\cos \phi_B^{(1)}, \sin \phi_B^{(1)}, 0) \\ \mathbf{B}^{(2)} &= B_0^{(2)} \cos \omega t \sin \eta\omega t (\cos \phi_B^{(2)}, \sin \phi_B^{(2)}, 0). \end{aligned} \quad (3.8)$$

As before we suppose that there is no overlap between the fields and that the time averages of the electric and magnetic energies are equal and equally distributed between the two circuits.

The quadrupole moment for the configuration (3.8) can be written as

$$\Pi(t) = \Pi^{(1)}(t) + \Pi^{(2)}(t) \quad (3.9)$$

where

$$\begin{aligned} \Pi^{(1)}(t) &= \cos^2(\eta\omega t) [\Pi_0^{(1)} + \Pi_{2\omega}^{(1)} \cos(2\omega t)] \\ \Pi^{(2)}(t) &= \sin^2(\eta\omega t) [\Pi_0^{(2)} + \Pi_{2\omega}^{(2)} \cos(2\omega t)]. \end{aligned} \quad (3.10)$$

The quadrupole moment of each circuit is of the form (3.3) multiplied by a low-frequency modulating factor. To obtain the maximum efficiency at the resonance with the beat frequency  $2\eta\omega$ , we choose the relative orientation of the electric and magnetic fields in each circuit in such a way that  $\Pi_{2\omega}^{(1)} = \Pi_{2\omega}^{(2)} = 0$ . This implies (see equation (3.4))

$$\phi_E^{(1)} - \phi_B^{(1)} = \phi_E^{(2)} - \phi_B^{(2)} = n\pi. \quad (3.11)$$

We can thus rewrite equation (3.9) as:

$$\Pi(t) = \Pi_0 + \Pi_{2\eta\omega} \cos(2\eta\omega t) \quad (3.12)$$

where

$$\Pi_0 = \frac{1}{2}(\Pi_0^{(1)} + \Pi_0^{(2)}) \quad \Pi_{2\eta\omega} = \frac{1}{2}(\Pi_0^{(1)} - \Pi_0^{(2)}). \quad (3.13)$$

By the same argument used before we recognise that the maximum value of  $|\Pi_{2\eta\omega}|$  obtains when

$$\phi_E^{(1)} - \phi_E^{(2)} = \pm \pi/2 \quad (3.14)$$

i.e. when the electric fields in two circuits are at right angles. The geometry corresponding to conditions (3.11) and (3.14) is sketched in figure 2.

A more practical high-frequency detector can be realised with two cavities at right angles coupled in such a way that the EM energy oscillates between them at the frequency of the GW.

The calculation of  $\Pi$  is in this case more complicated since the EM fields inside the cavity cannot be described by the simple formula (3.8).

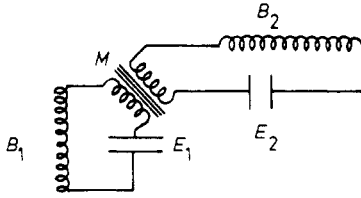


Figure 2. High-frequency LC resonator.

In terms of the EM quality factor  $Q_{EM} = \omega/\gamma_{EM}$  the energy transfer (equation (2.24)) can be written for both the low- and the high-frequency detectors in the form

$$\Delta W = h\rho \frac{\Omega}{\omega} Q_{EM} W_0(1 - 2\lambda). \tag{3.15}$$

For the low-frequency detector,  $\Omega/\omega = 2$ , whereas the high-frequency one has  $\Omega/\omega = 2\eta \ll 1$ . In the latter case the energy transfer  $\Delta W$  can be split into the sum of two terms

$$\Delta W = \Delta W_+ + \Delta W_- \tag{3.16}$$

representing the energy transferred by the GW to the two eigenmodes of the detector with frequencies  $\omega_+, \omega_-$  respectively. It can be shown that

$$\frac{d}{dt} \left( \frac{\Delta W_+}{\omega_+} + \frac{\Delta W_-}{\omega_-} \right) = (\text{oscillating terms}) \tag{3.17}$$

where the oscillating terms are of order  $h$  and have frequencies much larger than  $\Omega$ . From equations (3.15) and (3.17) it follows that:

$$|\Delta W_+| \cong |\Delta W_-| \cong \frac{1}{\eta} |\Delta W|. \tag{3.18}$$

Therefore the energy transferred to each eigenmode is much larger than the total energy transfer.

We remark that also the microwave detector suggested by Braginskii and Menskii (1971) can be described in terms of equation (2.15). Indeed one can show that the quadrupole moment  $\Pi$  of a wave packet with wavenumber  $\mathbf{k} = \mathbf{k}(t)$  and central frequency  $\omega$  is of the form

$$\Pi_{ij} = k_i k_j / \omega^2. \tag{3.19}$$

The energy  $W$  of the wave packet is

$$W = \omega N \tag{3.20}$$

where  $N$  is the number of photons in the packet. In the geometrical optics approximation  $N$  is an adiabatic invariant and therefore equation (2.15), in the absence of absorption, reduces to

$$\frac{d\omega}{dt} = \omega \text{Tr} \left( \frac{\dot{\mathbf{k}} \otimes \mathbf{k}}{\omega^2} \right). \tag{3.21}$$

The effect of the gravitational wave is in this case to change the average frequency of the wave packet. Equation (3.21) can be shown to follow directly from the geodesic equation for a light ray.

#### 4. Conclusion

In this paper we have analysed the principle of a detector whose response is linear in  $h$ . The advantage of EM detectors is that they are easily tunable over a wide range of frequencies. Since, to increase their sensitivity one is forced to use very high  $Q$ -values, the bandwidth of these detectors is correspondingly small. They are therefore ideal for studying monochromatic sources of known frequency.

A limitation of a detector linear in  $h$  is the difficulty in measuring  $\Delta W/W$  in the presence of a large energy in the two eigenmodes. With the present technology one can hope to achieve an accuracy of  $10^{-6}$  in  $\Delta W/W$ . With a  $Q$ -value of  $10^{11}$  which has already been achieved this leads to a minimum detectable  $h \sim 10^{-17}$ , of the same order of magnitude as the one obtained with mechanical detectors. An improvement in  $\Omega$  by two orders of magnitude would make this detector competitive with the mechanical ones now under study (see e.g. Drever 1977).

The same kind of EM detector discussed in this paper can be operated in a different way by loading the cavity in only one of the two eigenmodes. In this case the energy  $\Delta W_+$  transferred to the empty eigenmode is proportional to  $h^2 Q_{EM}^2 W$ . The advantage of this method of operation is that the minimum detectable energy  $\Delta W_+$  can be as small as  $10^{-9}$  erg. With this sensitivity and with an energy  $W_- \sim 10^7$  erg one can reach  $h = 10^{-19} - 10^{-21}$  with  $Q_{EM} = 10^{11} - 10^{13}$  respectively. Only a detailed analysis of the noise and of the feasibility of the detector will decide whether this sensitivity in  $h$  can actually be achieved.

#### Acknowledgments

We wish to thank Dr C Rubbia for his active contribution during the early stages of this work, and Dr Ph Bernard for his continuous interest which has helped us to understand many subtle problems connected with the operation of microwave cavities. We thank Drs J Bell, B Ferretti, F Pacini and J A Wheeler for several illuminating discussions and for calling our attention to the problem of the motion of the detector's walls. We are indebted to Mr C M Caves for sending us a manuscript where the problem of the wall's motion is discussed in a way similar to ours.

Finally one of us (FP) is grateful to CERN for the hospitality extended to him.

#### Appendix

If one takes into account the finite size of the detector, the correction  $h_{\mu\nu}(\lambda)$  produced by the GW takes the form

$$h_{\mu\nu}(\lambda) = (2\lambda - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \lambda [x^i(\lambda) \ddot{a}_{ij} x^j(\lambda)] \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad (A.1)$$

where  $i, j = 1, 2$  and  $a_{ij}$  is defined by equation (2.3).

The first term in the right-hand side of equation (A.1) is of order  $h$ , whereas the second is of order  $h(\Omega L)^2$ , here  $L$  is the characteristic size of the detector. The latter has been neglected in equation (2.2).

The coordinates  $x^\mu(\lambda_1)$  and  $x^\mu(\lambda_2)$  are related, to first order in  $(\lambda_2 - \lambda_1)h$ , by the transformations:

$$\begin{aligned} x^0(\lambda_2) &= x^0(\lambda_1) + \frac{1}{2}(\lambda_2 - \lambda_1)x^i \dot{a}_{ij}x^j \\ x^i(\lambda_2) &= x^i(\lambda_1) - (\lambda_2 - \lambda_1)a^i_j x^j \\ x^3(\lambda_2) &= x^3(\lambda_1) + \frac{1}{2}(\lambda_2 - \lambda_1)x^i \dot{a}_{ij}x^j \end{aligned} \tag{A.2}$$

where  $a^i_j = \eta^{ik}a_{kj}$ . To the same approximation there is no need to distinguish in the second term in the right-hand side of equation (2.5) between  $x^i(\lambda_1)$  and  $x^i(\lambda_2)$ . With the same argument we can drop the label  $\lambda$  in the term  $x^i(\lambda)\ddot{a}_{ij}x^j(\lambda)$  in equation (A.1). We remark that in equations (A.2) the second term in the equations for  $x^0$  and  $x^3$  is of order  $h\Omega L$ , whereas in the equation for  $x^i$  the corresponding term is of order  $h$ .

For every  $\lambda$  the geodesic equation of the centre of mass is  $x^i = 0, i = 1, 2, x^3 = 0$ . The square of the spatial distance between two particles in the  $(x^1, x^2)$  plane with coordinates  $x^i$  and  $x^i + \Delta x^i$  is:

$$\Delta l^2 = \Delta x^i [\eta_{ij} + (2\lambda - 1)a_{ij}] \Delta x^j. \tag{A.3}$$

Due to the time dependence of  $a_{ij}$ , even if two particles are at rest, their spatial distance varies with time for all  $\lambda$ 's different from  $\frac{1}{2}$ .

Two values of  $\lambda$  are of special significance:

(a)  $\lambda = 0$ . In this case a free particle at rest in the absence of the GW, remains at rest even when the GW is turned on. This means that the system  $\lambda = 0$  is the free-falling system;  $g_{\mu\nu}(\lambda = 0)$  is thus the metric tensor in the transverse traceless gauge.

(b)  $\lambda = \frac{1}{2}$ . In this case  $h_{\mu\nu}$  is of order  $h(\Omega L)^2$  and the space distance  $h_{\mu\nu}$  between two points at rest in the  $(x^1, x^2)$  plane is constant. This is the proper reference frame (Misner *et al* 1973) attached to an infinitely rigid detector. In this coordinate frame a free particle, which would be at rest in the absence of the GW, appears to oscillate.

Let us now consider an idealised detector whose walls are formed by a system of equal particles elastically bound to the centre of mass. The system in which the 'walls' of this detector are at rest will be called its rest system and the corresponding value of  $\lambda$  depends upon the mechanical properties of the detector. In the limit of vanishing binding we have  $\lambda = 0$ , whereas for an infinitely rigid binding we have  $\lambda = \frac{1}{2}$ . To find the value of  $\lambda$  appropriate to an oscillator with frequency  $\omega_m$  we consider a particle of mass  $m$ , elastically bound to the centre of mass of the detector, moving under the influence of the GW. We assume that in the absence of the GW the particle is at rest in the  $(x^1, x^2)$ -plane at  $x^i = X^i$ . Let  $l^2$  be the spatial distance from the origin as defined in equation (A.3) of a point  $x^i$  and let  $l^i(x)$  be defined by

$$l^i(x)\eta_{ij}l^j(x) = l^2(x). \tag{A.4}$$

Then, to order  $h$ ,

$$l^i(x) = x^i + \frac{2\lambda - 1}{2} a^i_j x^j \tag{A.5}$$

and

$$l^i(x^i = X^i, h = 0) = X^i. \tag{A.6}$$

Under the action of the GW the particle is displaced by an amount  $\Delta l^i = l^i - X^i$  of

order  $h$ . The restoring force is  $-\omega_m^2 \Delta l^i$ , where  $\omega_m$  is the frequency of the oscillator and

$$\Delta l^i = \xi^i + \frac{2\lambda - 1}{2} a^i_j X^j, \quad \xi^i = x^i - X^i. \quad (\text{A.7})$$

The displacement  $\Delta l^i$  is the sum of two terms: the first one represents the difference between the coordinate of the particle and that of the equilibrium position; the second is due to the change of the distance of the equilibrium position  $X^i$  from the centre of mass. By neglecting terms of order  $hv$ , where  $v$  is the velocity of the particle under the influence of the GW, the equation of motion for the particle reads:

$$\ddot{\xi}^i = A^i - \omega_m^2 \Delta l^i \quad (\text{A.8})$$

where

$$A^i = -\Gamma^i_{00}|_{x^i=X^i} = -\lambda \ddot{a}^i_j X^j \quad (\text{A.9})$$

and  $\Gamma^i_{00}$  is a Christoffel index. Then equation (A.8) becomes:

$$\ddot{\xi}^i + \omega_m^2 \xi^i = [\lambda \Omega^2 - (\lambda - \frac{1}{2}) \omega_m^2] a^i_j X^j. \quad (\text{A.10})$$

If  $\omega_m \neq \Omega$ , the right-hand side of equation (A.10) vanishes for

$$\lambda = \frac{1}{2} \frac{\omega_m^2}{\omega_m^2 - \Omega^2} \quad (\text{A.11})$$

and therefore in the system with  $\lambda$  given by (A.11) a particle which was originally at rest remains at rest even in the presence of the GW. To determine the value of  $\lambda$  which transforms away the forced motion due to the GW when  $\omega_m = \Omega$ , one must introduce in the right-hand side of equation (A.8) a damping term  $-2\gamma_m \Delta l^i$  and generalise the above discussion to the case where  $\lambda$  is complex. The absolute value of  $\lambda$  at resonance can easily be seen to be

$$|\lambda| \approx \frac{\omega_m}{\gamma_m} = Q_m. \quad (\text{A.12})$$

Due to the finite size of the detector the relations between  $(E, B)$  and  $(D, H)$  are also modified. Instead of equations (2.9) and (2.10) we get

$$\begin{aligned} D^i &= -\eta^{ij} (1 + \frac{1}{2} x^k \ddot{a}_{kl} x^l) E_j - \lambda x^k \ddot{a}_{kl} x^k \ddot{a}_{kl} x^l \sigma^{ij} B_j - (1 - 2\lambda) a^{ij} E_j \\ D^3 &= (1 - \frac{1}{2} \lambda x^k \ddot{a}_{kl} x^l) E_3 \\ H^i &= -\eta^{ij} (1 - \frac{3}{2} x^k \ddot{a}_{kl} x^l) B_j - \lambda x^k \ddot{a}_{kl} x^k \ddot{a}_{kl} x^l \chi^{ij} E_j + (1 - 2\lambda) a^{ij} B_j \\ H^3 &= (1 - \frac{1}{2} \lambda x^k \ddot{a}_{kl} x^l) B_3 \quad i, j, k, l = 1, 2 \end{aligned} \quad (\text{A.13})$$

where

$$\sigma^{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.14})$$

and

$$a^{ij} = \eta^{ik} a_{kl} \eta^{lj}. \quad (\text{A.15})$$

We remark that equations (2.10) are obtained from (A.13) by neglecting terms of order  $h(\Omega L)^2$ .

**References**

- Amaldi E and Pizzella G 1978 *Consiglio Nazionale delle Ricerche, Frascati Report LPS-78-3*,  
Billing H, Kafka P, Maischberger K, Meyer F and Winkler W 1975 *Lett. Nuovo Cim.* **12** 111  
Braginskii V B, Caves C M and Thorne K S 1977 *Phys. Rev. D* **15** 2047  
Braginskii V B, Grishchuk I D, Doroshkevich A G, Zel'dovich Ya B, Novikov I D and Zazhin M V 1974  
*Sov. Phys.-JETP* **38** 865  
Braginskii V B, Manukin A B, Popov E I, Rudenko V N and Khorev A A 1972 *JETP Lett.* **16** 108  
Braginskii V B and Menskii M B 1971 *JETP Lett.* **13** 417  
Bramanti D and Maischberger K 1972 *Lett. Nuovo Cim.* **4** 1007  
Drever R W P 1977 *Q. J. R. Astron. Soc.* **18** 9  
Drever R W P, Hough J, Bland R and Lessnoff G W 1973 *Nature* **246** 340  
Hirakawa H and Narihara K 1975 *Phys. Rev. Lett.* **35** 330  
Landau L D and Lifshitz E M 1962 *The Classical Theory of Fields* (Oxford: Pergamon Press) p 296  
Levine J L and Garwin R L 1973 *Phys. Rev. Lett.* **31** 173  
Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman) p 327  
Press W H and Thorne K S 1972 *A. Rev. Astron. Astrophys.* **10** 335  
Tyson J A 1973 *Phys. Rev. Lett.* **31** 326  
Weber J 1969 *Phys. Rev. Lett.* **22** 1320